Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **97**, 1-5; **http://www.math.u-szeged.hu/ejqtde/**

CORRIGENDUM TO ON A CLASS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH INFINITE DELAY

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Abstract

This paper serves as a corrigendum to the paper titled On a class of differential-algebraic equations with infinite delay appearing in EJQTDE no. 81, 2011. We present here a corrected version of Lemma 5.5 and Corollary 5.7.

1 Introduction

In Section 5 of [1] we investigated examples of applications of that paper's results to a particular class of implicit differential equations. For so doing we used a technical lemma from linear algebra that, unfortunately, turns out to be flawed. As briefly discussed below this affects only marginally our paper's results (just a corollary in Section 5 of [1]).

The simple example below shows that there is something wrong with Lemma 5.5 in [1]. In the next section we provide an amended version of this result.

Example 1.1. *Consider the matrices*

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

2000 Mathematics Subject Classifications: 34A09, 34K13, 34C40
Keywords and phrases: differential-algebraic equations, retarded functional equations, periodic solution, periodic perturbation, infinite delay

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Clearly, $\ker C^T = \ker E^T = \mathrm{span}\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ *for all* $t \in \mathbb{R}$*. The matrices*

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

realize a singular value decomposition for E*. Nevertheless*

$$P^T CQ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which is not the form expected from Lemma 5.5 in [1]. The problem, as it turns out, is that $\ker C \neq \ker E$*.*

Luckily, the impact of the wrong statement of [1, Lemma 5.5] on [1] is minor: all results and examples (besides Lemma 5.5, of course) remain correct, with the exception of Corollary 5.7 where it is necessary to assume the following further hypothesis:

$$\ker\, C(t) = \ker\, E,\ \forall\, t \in \mathbb{R}.$$

(A corrected statement of Corollary 5.7 of [1] can be found in the next section, Corollary 2.2.)

2 Corrected Lemma and its consequences

We present here a corrected version of Lemma 5.5 in [1].

Lemma 2.1. *Let* $E \in \mathbb{R}^{n\times n}$ *and* $C \in C\big(\mathbb{R}, \mathbb{R}^{n\times n}\big)$ *be respectively a matrix and a matrix-valued function such that*

$$\ker\, C^T(t) = \ker\, E^T,\ \forall\, t \in \mathbb{R},\ \textit{and}\ \dim\ker\, E^T > 0, \tag{2.1}$$

Put $r = \mathrm{rank}\ E$*, and let* $P, Q \in \mathbb{R}^{n\times n}$ *be orthogonal matrices that realize a singular value decomposition for* E*. Then it follows that*

$$P^T C(t) Q = \begin{pmatrix} \widetilde{C}_{11}(t) & \widetilde{C}_{12}(t) \\ 0 & 0 \end{pmatrix}, \quad \forall t \in \mathbb{R}, \tag{2.2}$$

with $\widetilde{C}_{11} \in C\big(\mathbb{R}, \mathbb{R}^{r\times r}\big)$ *and* $\widetilde{C}_{12} \in C\big(\mathbb{R}, \mathbb{R}^{r\times n}\big)$*.*

If, furthermore,

$$\ker\, C(t) = \ker\, E,\ \forall\, t \in \mathbb{R}, \tag{2.3}$$

then $\widetilde{C}_{12}(t) \equiv 0$*. Namely, in this case,*

$$P^T C(t) Q = \begin{pmatrix} \widetilde{C}_{11}(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall t \in \mathbb{R}, \tag{2.4}$$

with $\widetilde{C}_{11}(t)$ *nonsingular for all* $t \in \mathbb{R}$*.*

Proof. Our proof is essentially a singular value decomposition (see, e.g., [2]) argument, based on a technical result from [3].

Observe that (2.1) imply $\operatorname{rank} E = \operatorname{rank} C(t) = r > 0$ for all $t \in \mathbb{R}$. In fact,

$$\begin{aligned}\operatorname{rank} E = \operatorname{rank} E^T = n - \dim\ker E^T = \\ = n - \dim\ker C(t)^T = \operatorname{rank} C(t)^T = \operatorname{rank} C(t).\end{aligned}$$

Since $\operatorname{rank} C(t)$ is constantly equal to $r > 0$, by inspection of the proof of Theorem 3.9 of [3, Chapter 3, §1] we get the existence of orthogonal matrix-valued functions $U, V \in C\big(\mathbb{R}, \mathbb{R}^{n\times n}\big)$ and $C_r \in C(\mathbb{R}, \mathbb{R}^{r\times r})$ such that, for all $t \in \mathbb{R}$, $\det C_r(t) \neq 0$ and

$$U^T(t)C(t)V(t) = \begin{pmatrix} C_r(t) & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.5}$$

Let $U_r, V_r \in C\big(\mathbb{R}, \mathbb{R}^{n\times r}\big)$ and $U_0, V_0 \in C(\mathbb{R}, \mathbb{R}^{n\times(n-r)})$ be matrix-valued functions formed, respectively, by the first r and $n-r$ columns of U and V. An argument involving Equation (2.5) shows that, for all $t \in \mathbb{R}$, the space $\operatorname{im} C(t)$ is spanned by the columns of $U_r(t)$. Also, (2.5) imply that the columns of $V_0(t)$, $t \in \mathbb{R}$, belong to $\ker C(t)$ for all $t \in \mathbb{R}$. A dimensional argument shows that they constitute a basis $\ker C(t)$. Analogously, transposing (2.5), we see that the columns of $V_r(t)$ and $U_0(t)$ are bases of $\operatorname{im} C(t)^T$ and $\ker C(t)^T$ respectively.[1]

Let now P_r, Q_r and P_0, Q_0 be the matrices formed taking the first r and $n-r$ columns of P and Q, respectively. Since P and Q realize a singular value decomposition of E, proceeding as above one can check that the columns of P_r, Q_r, P_0 and Q_0 span $\operatorname{im} E$, $\operatorname{im} E^T$, $\ker E^T$, and $\ker E$, respectively.

We claim that $P_0^T U_r(t)$ is constantly the null matrix in $\mathbb{R}^{(n-r)\times r}$. To prove this, it is enough to show that for all $t \in \mathbb{R}$, the columns of P_0 are orthogonal to those of $U_r(t)$. Let v and $u(t)$, $t \in \mathbb{R}$, be any column of P_0 and of $U_r(t)$, respectively. Since for all $t \in \mathbb{R}$ the columns of $U_r(t)$ are in $\operatorname{im} C(t)$, there is a vector $w(t) \in \mathbb{R}^n$ with the property that $u(t) = C(t)w(t)$, and

$$\langle v, u(t)\rangle = \langle v, C(t)w(t)\rangle = \langle C(t)^T v, w(t)\rangle = 0, \quad \forall t \in \mathbb{R},$$

because $v \in \ker E^T = \ker C(t)^T$ for all $t \in \mathbb{R}$. This proves the claim. A similar argument shows that $P_r^T U_0(t)$ is identically zero as well.

[1] In fact, the orthogonality of the matrices $V(t)$ and $U(t)$ for all $t \in \mathbb{R}$, imply that the columns of $U_r(t)$, $V_r(t)$, $U_0(t)$ and $V_0(t)$ are respective orthogonal bases of the spaces $\operatorname{im} C(t)$, $\operatorname{im} C(t)^T$, $\ker C(t)^T$ and $\ker C(t)$.

Since for all $t \in \mathbb{R}$

$$P^T U(t) = \begin{pmatrix} P_r^T U_r(t) & 0 \\ 0 & P_0^T U_0(t) \end{pmatrix}$$

is nonsingular, we deduce in particular that so is $P_r^T U_r(t)$.

Let us compute the matrix product $P^T C(t) Q$ for all $t \in \mathbb{R}$. We omit here, for the sake of simplicity, the explicit dependence on t.

$$\begin{aligned} P^T C Q = P^T U U^T C V V^T Q &= \begin{pmatrix} P_r^T U_r & 0 \\ 0 & P_0^T U_0 \end{pmatrix} \begin{pmatrix} C_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_r^T Q_r & V_r^T Q_0 \\ V_0^T Q_r & V_0^T Q_0 \end{pmatrix} \\ &= \begin{pmatrix} P_r^T U_r C_r V_r^T Q_r & P_r^T U_r C_r V_r^T Q_0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

which proves (2.2).

Let us now assume that also (2.3) holds. We claim that in this case $V_0^T Q_r$ is identically zero. To see this we proceed as done above for the products $P_0^T U_r$ and $P_r^T U_0$. Let $v(t)$, $t \in \mathbb{R}$, be any column of $V_0(t)$, hence a vector of $\ker C(t)$ for all $t \in \mathbb{R}$, and let q be a column of $Q_r(t)$. Since the columns of Q_r lie in $\operatorname{im} E^T$, there is a vector $\ell \in \mathbb{R}^n$ with the property that $q = E^T \ell$, and

$$\langle v(t), q \rangle = \langle v(t), E^T \ell \rangle = \langle E v(t), \ell \rangle = 0, \quad \forall t \in \mathbb{R},$$

because $v(t) \in \ker C(t) = \ker E$ for all $t \in \mathbb{R}$. This proves the claim. A similar argument shows that $V_r^T Q_0(t)$ is identically zero as well. Hence,

$$V(t)^T Q = \begin{pmatrix} V_r(t)^T Q_r & 0 \\ 0 & V_0(t)^T Q_0 \end{pmatrix}$$

thus $V_r^T(t) Q_0$, and $V_0^T(t) Q_r$ are nonsingular. Also, plugging $V_0^T Q_r = 0$ in the above expression for $P^T C Q$ one gets (we omit again the explicit dependence on t)

$$P^T C Q = \begin{pmatrix} P_r^T U_r C_r V_r^T Q_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.6}$$

Which proves the assertion because $P_r^T U_r$, C_r, and $V_r^T Q_r$ are nonsingular. $\square$

In view of the corrected version of the above lemma, the statement of Corollary 5.7 of [1] can be rewritten as follows:

Corollary 2.2. *Consider Equation*

$$E \dot{\mathbf{x}}(t) = \mathcal{F}\big(\mathbf{x}(t)\big) + \lambda C(t) S(\mathbf{x}_t), \tag{2.7}$$

where the maps $C\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ *and* $S\colon BU\big((-\infty,0],\mathbb{R}^n\big) \to \mathbb{R}^n$ *are continuous,* E *is a (constant)* $n \times n$ *matrix,* $\mathcal{F}$ *is locally Lipschitz and* S *verifies condition (**K**) in [1]. Suppose also that* C *and* E *satisfy* (2.1) *and* (2.3)*, and that* C *is* T*-periodic. Let* $r > 0$ *be the rank of* E *and assume that there exists an orthogonal basis of* $\mathbb{R}^n \simeq \mathbb{R}^r \times \mathbb{R}^{n-r}$ *such that* E *has the form*

$$E \simeq \begin{pmatrix} E_{11} & E_{12} \\ 0 & 0 \end{pmatrix}, \text{ with } E_{11} \in \mathbb{R}^{r\times r} \text{ invertible and } E_{12} \in \mathbb{R}^{r\times(n-r)}.$$

Assume also that, relatively to this decomposition of $\mathbb{R}^n$*,* $\partial_2\mathcal{F}_2(\xi,\eta)$ *is invertible for all* $x = (\xi,\eta) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$*.*

Let Ω *be an open subset of* $[0,+\infty) \times C_T(\mathbb{R}^n)$ *and suppose that* $\deg(\mathcal{F}, \Omega \cap \mathbb{R}^n)$ *is well-defined and nonzero. Then, there exists a connected subset* Γ *of nontrivial* T*-periodic pairs for* (2.7) *whose closure in* Ω *is noncompact and meets the set* $\big\{(0,\overline{\mathbf{p}}) \in \Omega : \mathcal{F}(\mathbf{p}) = 0\big\}$*.*

This result follows as in [1] taking into account the modified version of the lemma.

References

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(Received November 22, 2012)